Math 253
Spring 2021

Name: Key

Quiz 5.3, 5.4

1. (5 points) What does the **integral test** tell you about the following series? $\sum_{n=3}^{\infty} ne^{-n^2}$ Make sure to check off the relevant hypotheses. If the integral test does not allow you to make a conclusion, state "no conclusion."

Let $f(x) = xe^{-x^2} \geq 0$ ✓

$f(x)$ is continuous ✓

$f(x)$ is decreasing because $f'(x) = e^{-x^2} \cdot xe^{-x^2} \cdot (-2x)$

$= e^{-x^2}(1-2x^2)$

< 0 for $x \geq 1$

$f(x) = ne^{-n^2}$ ✓

$$\int_3^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty} \int_3^t xe^{-x^2}\,dx = \lim_{t\to\infty} \int_{-9}^{-t^2} e^u \frac{du}{-2} = \lim_{t\to\infty} \frac{-e^u}{2}\Big|_{-9}^{-t^2}$$

$u = -x^2$

$du = -2x\,dx$

$$= \lim_{t\to\infty} \frac{-e^{-t^2}}{2} + \frac{e^{-9}}{2} = \frac{e^{-9}}{2}$$

Since $\int_3^{\infty} xe^{-x^2}\,dx$ converges, so does $\sum_{n=3}^{\infty} ne^{-n^2}$

2. (5 points) Consider the sequences defined by $a_n = \frac{n}{2^n}$ for $n \geqslant 1$ and $b_n = \frac{1}{2^n}$ for $n \geqslant 1$.

(a) How does a_n compare to b_n? I.e. is $a_n \geqslant b_n$, $a_n \leqslant b_n$, or neither?

$$\frac{n}{2^n} \geqslant \frac{1}{2^n}$$

(b) What is $\sum_{n=1}^{\infty} b_n$?

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$$
$$= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = 1$$

(c) What does the **Direct Comparison Test**—when applied to a_n and b_n—say (if anything) about whether or not $\sum_{n=1}^{\infty} a_n$ converges? If the direct comparison test does not allow you to make a conclusion for this choice of a_n and b_n, state "no conclusion."

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \geqslant \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

$\Rightarrow$ No conclusion